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Constructing some spectra

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We start from the following

<u>Problem 1</u>. Given integers $n, a > 0$, does there exist spectrum $E = E_p$ for each prime p and a constant c (independent of p) satisfying the followings ?

$$\# \text{ of generators of } \sum_{* < ap^n} \pi_*(E) \otimes Z_p < c$$

and

$$\# \text{ of generators of } \sum_{* < ap^n} H_*(E) \otimes Z_p < c.$$

As is easily seen the spectra $E = S^0$, $H_p = K(Z_p)$, BP do not give any answer of the above problem. However, for the case $n = 1$, an easy answer is given by taking $E = V(1)$ or the following $H_p(1)$. We define

$$H_p(1) = \text{fibre of } P^1 : H_p \longrightarrow \Sigma^q H_p = R_1^0 . H_p$$

and

$$BP(1) = \text{fibre of } r_1 : BP \longrightarrow \Sigma^q BP = R_1^0 . BP,$$

where $q = 2(p-1)$, R_i^j is a symbol having bidegree $(-1, 2p^j(p^i-1))$, r_A is the Landweber-Novikov operation and P^A is a cohomology operation dual to that of Milnor's (having a similar action to homolgy to r_A). Then $\pi_*(H_p(1))$ has only two generators 1 and $h = \{R_1^0\}$ of degree 0 and q-1,and

$$H_*(BP(1);\, Z_p) \;\cong\; Z_p[\xi_1^p,\xi_2,\xi_3,\ldots] \;\otimes\; \Lambda(\psi_1^0),$$

$$H_*(H_p(1);\, Z_p) \;\cong\; H_*(BP(1);\, Z_p) \;\otimes\; \Lambda(\tau_0,\tau_1',\tau_2',\ldots)$$

where $H_*(H_p;\, Z_p) = Z_p[\xi_1,\xi_2,\ldots] \otimes \Lambda(\tau_0,\tau_1,\ldots)$ by Milnor, $\tau_i' = \tau_i - \tau_0\xi_i$, $\psi_1^0 = [R_1^0.\xi_1^{p-1}]$ of degree pq-1 and $H_*(BP;\, Z_p) = Z_p[\xi_1,\xi_2,\ldots]$ for the mod p reduction ξ_i of $m_i \in H_*(BP) = Z_{(p)}[m_1,m_2,\ldots]$, $\deg \xi_i = \deg m_i = 2(p^i-1)$, $\deg \tau_i = 2p^i-1$. The element ψ_1^0 is detected by the secondary operation Ψ_1^0 associated with the relation $P^{p-1}P^1 = 0$. Taking p sufficiently large ($pq-2 \geq ap$) we see that $H_p(1)$ gives an answer to the problem for n = 1. Also we may regard that the spectrum V(1) is the (pq-2)-skeleton of $H_p(1)$ for $p \geq 3$.

Now our problem is to costruct spectra of such a sort $H_p(n)$, BP(n) for n = 2,3,... . We consider

Problem 2. Given positive integer n, can we construct a chain complex C(n) satisfying the following conditions ? For $X = H_p$ or BP

$$C(n) = \Lambda(R_i^j\ ;\ i+j \leq n) \otimes X^*(X),$$

for monomials x, y in $\Lambda(R_j^i)$, $\partial(x \otimes 1) = \sum y \otimes f_{x,y}$

$$f_{x,y} \equiv \begin{cases} -1 & \text{if } x = zR_i^j \text{ and } y = zR_k^jR_{i-k}^{j+k} \\ P_i^j \text{ or } r_i^j & \text{if } y = xR_i^j\ ;\ P_i^j = P^A,\ r_i^j = r_A \text{ for } A = p^j\Delta_i, \\ 0 & \text{otherwise} \end{cases}$$

modulo p and higher terms in the subalgebra generated by r_i^j's.

The chain complex is represented by a sequence

$$C(n) : \quad X_0 = X \xrightarrow{\partial_0} X_1 \xrightarrow{\partial_1} X_2 \longrightarrow \cdots \longrightarrow X_{\binom{n+1}{2}} = X$$

where X_r is the product (wedge) of $x.X$ for monomials x in $\Lambda(R_i^j)$ of the length r.

We denote by $H_p(n)$ resp. $BP(n)$ a fibre (tower, relaization or desuspension of iterated cones) of the above sequence $C(n)$ if it exists.

Lemma (i) Assume the existence of $C(n)$. Then there exists $BP(n)$ for $p \geq \frac{1}{4}(n^2 + n + 2)$ and $H_p(n)$ for $p \geq \frac{1}{4}(n^2 + 3n + 4)$ or $p = 3$, $n = 2$. They are unique if the inequalities hold.

(ii) Assume the existence of $BP(n)$. Then there exists a spectral sequence : $E_2 = H_*(C(n); Y_*(BP)) \Longrightarrow Y_*(BP(n))$, which collapses if $p > \frac{1}{4}(n^2 + n + 2)$ and if $Y = S, H, H_p, V(m)$ or $= BP$. Similar spectral sequence exists for $H_p(n)$.

In general, a fibre of C(n) exists if $[\Sigma^{k-2}X_r, X_{r+k}] = 0$ for $k \geq 3$. Then the lemma is proved by the fact $BP^*(BP) = 0$ for $* \not\equiv 0 \pmod q$ and also counting the number of Bocksteins in the monomials of $H_p(H_p) = H_*(H_p; Z_p)$ of appropriate degrees.

Corollary. $\pi_*(H_p(n)) = H_*(C(n); Z_p)$, so the number of the generators of $\pi_*(H_p(n))$ is not greater than $2^{\binom{n+1}{2}}$.

In $H_*(BP; Z_p)$ r_i^j acts same as P_i^j Consier the subalgebra P^* of the mod p Steenrod algebra $H_p^*(H_p)$ sapnned by P^A, then the associated graded algebra $E^0(P^*)$ is the envelopping algebra over a Lie algebra mod p spanned by P_i^j with the relation $[P_k^j, P_{i-k}^{j+k}] = P_i^j$. So, modulo p and higher terms, C(n) changes to May's resolution of (non-restricted) Lie algebra $\{P_i^j\}$, and we have

Lemma. There exist spectral sequences : $'E_2 = Z_p[\xi_1^{p^n},\ldots,\xi_n^p,\xi_{n+1},\ldots] \otimes \Lambda(\psi_i^j;\ i+j \leq n) \Longrightarrow H_*(C(n); H_*(BP; Z_p))$ and $''E_2 = {}'E_2 \otimes \Lambda(\tau_0,\tau_1',\tau_2',\ldots) \Longrightarrow H_*(C(n); H_*(H_p; Z_p))$, where $\deg \psi_i^j = 2p^{j+1}(p^i-1)-1$.

Corollary. If C(n) exists then Problem 1 is affirmative by $H_p(n)$.

Note 1. C(n) may be regarded as a sort of (unusual) resolution of $H_p(n)$ or BP(n).

Note 2. Let p be sufficiently large w. r. t. n. If $H_p(n)$ exists and the above associated spectral sequences collapse, then we define a spectrum VB(n) as the $(p^n q-2)$-skeleton of $H_p(n)$:

$H_*(VB(n); Z_p) \cong \Lambda(\tau_0, \tau'_1, \tau'_2, \dots, \tau'_n) \otimes \Lambda(\psi^j_i;\ i+j < n)$.

Similarly, for the $(p^n q-2)$-skeleton $B(\binom{n}{2})$ of $BP(n)$,

$H_*(B(\binom{n}{2}); Z_p) \cong \Lambda(\psi^j_i;\ i+j < n)$.

Note 3. If $V(n)$ and $B(\binom{n}{2})$ exist then $V(n) \wedge B(\binom{n}{2})$ may be regarded as $VB(n)$.

Now we can prove the following

Theorem. For $p \geq 3$, $BP(2)$, $H_p(2)$, $VB(2)$ and $B(3)$ exist. For $p \geq 5$, $BP(3)$, $H_p(3)$ and $VB(3)$ exist.

The main part of the proof is the construction of $C(2)$ and $C(3)$. If $C(n)$ is constructed for $X = BP$, it is also constructed for $X = H_p$ just by changing r^j_i by P^j_i. So we construct $C(2)$ and $C(3)$ for $X = BP$ only.

First consider the case $n = 2$. Since $r_1 = r^0_1$, $r_p = r^1_1$ and $r_{0,1} = r^0_2$ enjoy the relations

$[r_1, r_p] = r_{0,1}$ and $[r_1, r_{0,1}] = [r_p, r_{0,1}] = 0$,

$C(2)$ is defined by the formulas in Problem 2 without taking modulus, that is, it is represented by the following diagram (replacing $x.BP$ by x) :

$$\begin{tikzcd}[column sep=large]
 & R_1^0 \arrow[r, "-r_{0,1}"] \arrow[dr, "-r_p" near start] & R_2^0R_1^0 \arrow[dr, "r_p"] & \\
1 \arrow[ur, "r_1"] \arrow[r, "r_{0,1}"] \arrow[dr, "r_p"'] & R_2^0 \arrow[ur, "r_1" near start] \arrow[r, "-1"] \arrow[dr, "r_p" near start] & R_1^1R_1^0 \arrow[r, "-r_{0,1}"] & R_1^1R_2^0R_1^0 \\
 & R_1^1 \arrow[ur, "r_1" near start] \arrow[r, "r_{0,1}"'] & R_1^1R_2^0 \arrow[ur, "r_1"'] &
\end{tikzcd}$$

Note 4. $BP(1)$ is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_p e^{pq} \cup_{\alpha_1} e^{(p+1)q} \cup e^{2pq-1} \quad \cdots$$

where $P^p(S^0) = e^{pq}$ and $P^1(e^{pq}) = e^{(p+1)q}$ in mod p cohomology.

$BP(2)$ is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1\beta_1^{p-1}} e^{p^2q-1} \cup_p e^{p^2q} \cup e^{(p^2+p)q-2} \cup e^{(p^2+p)q-1} \quad \cdots$$

where $P^{p^2}(S^0) = e^{p^2q}$.

Note 5. There exists a chain map $g : C(2) \longrightarrow \Sigma^{pq}C(2)$ of degree 1 such as

$$\begin{tikzcd}
 & 1 \arrow[d, "0"'] & R_1^0 \arrow[dr, "r_{p-1}"'] & R_2^0 \arrow[d, "v_1"] & R_1^1 \arrow[dl, "-p"] & \cdots\cdots \\
g \arrow[d] & (\phi) & & 1 & & \cdots\cdots \\
{} & & & & & \cdots\cdots .
\end{tikzcd}$$

Moreover $gg = 0$. This induces maps $g : BP(2) \longrightarrow \Sigma^{pq-1}BP(2)$ and $g : H_p(2) \longrightarrow \Sigma^{pq-1}H_p(2)$ such that $g_*(\psi_1^0) = 1$.

Note 6. By considering a fibre of the sequence $H_p(2) \xrightarrow{g} \Sigma^{pq-1}H_p(2) \xrightarrow{g} \Sigma^{2(pq-1)}H_p(2) \longrightarrow \ldots$, we can obtain $V(2)$ for $p > 3$, since g kills ψ_1^0. But this breaks for $p = 3$ and we obtain $V(1\frac{1}{2})$ only.

Note 7. By a chain equivalence R_2^0 and $R_1^1R_1^0$ are cancelled, and we obtain an equivalent chain complex

$$C(2)' \ : \quad \begin{tikzcd} & h \arrow[rr, "-r_{0,1} - r_1r_p"] \arrow[drr, "r_pr_p"'] & & g \arrow[dr, "r_p"] & \\ 1 \arrow[ur, "r_1"] \arrow[dr, "r_p"'] & & & & hk. \\ & h_1 \arrow[rr, "r_{0,1} - r_pr_1"'] \arrow[urr, "r_1r_1"] & & k \arrow[ur, "r_1"'] & \end{tikzcd}$$

Moreover, combining with the reduced form of g, as in Note 6, we obtain a BP-resolution of S^0 up to degree $p^2q - 2$ which is essentially same as "BP-relation" of Thomas-Zahler used in proving the non-triviality of some γ_i.

Finally we consider the case $n = 3$. The operations $\{r_i^j;\ i+j \leq 3\}$ are no more closed under $[\ ,\ ]$:

$$[r_1, r_{p^2}] = r_{p^2-p}r_{0,1}, \quad [r_p, r_{p^2}] = r_{0,p} + fr_{0,1}, \quad [r_{0,1}, r_{p^2}] = r_{0,0,1},$$

$$[r_1, r_{0,p}] = r_{0,0,1}, \quad [r_p, r_{0,p}] = r_{p-1}r_{0,0,1}, \quad [r_{0,1}, r_{0,p}] = 0,$$

$$[r_{p^2}, r_{0,p}] = (r_{p^2-1} - r_{0,p-1})r_{0,0,1} \text{ and } [r_i^j, r_{0,0,1}] = 0 \text{ for } i+j \leq 3,$$

where f is uiquely determined by the second equality (explicitly : $f = - r_{p^2-1} + \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} r_{p-i-1} r_{p^2-pi-p} r_{0,i}$).

However we can construct $C(3)$ by taking $f_{x,y} = -1$ or r_i^j for the first two cases in Problem 2 and modifying $f_{x,y}$ for the third case as follows. If gr_m^n is a term of above $[r_i^j, r_k^\ell]$ then

$$f_{x,y} = -g \quad \text{for} \quad x = zR_m^n, \ y = zR_k^\ell R_i^j \quad (R_i^j, R_k^\ell, R_m^n \not\in z).$$

In order to complete the definition of   in $C(3)$, we must add two more extra cases :

$$f_{x,y} = \begin{cases} -r_{p^2-p-1} & \text{for } x = zR_3^0R_2^0, \ y = zR_1^2R_2^1R_1^0 \quad (x = 1, R_1^1) \\ -f' & \text{for } x = zR_3^0R_2^0, \ y = zR_1^2R_2^1R_1^1 \quad (x = 1, R_1^0), \end{cases}$$

where f' is determined by

$$f'r_{0,1} = [r_{p-1}, r_{p^2}] - [r_p, r_{p^2-1}].$$

To check the condition $= 0$, we need various relations in $[\ ,\ ]$. For example, between $R_3^0R_2^0$ and $R_1^2R_2^1R_1^1R_1^0$ there are 10 monomials connected by non-trivial maps. They are cancelled by $[1, f] = [1, r_{p^2-1}] = 0$ and $[f', r_1] = [r_{p-1}, r_{p^2-p}] + [r_p, r_{p^2-p-1}]$.

Consequently we can construct $C(3)$, and then $H_p(3)$ and $BP(3)$ for $p > 3$

by applying the first lemma. For $p = 3$ and $X = BP$, $C(3)$ is realized except the last term, then $B(3)$ is obtained as a skeleton.

Note 8. Let $B(2)$ be the $((p^2+p)q - 2)$-skeleton of $B(3)$ then there are cofiberings

$$S^0 \longrightarrow B(1) \longrightarrow S^{pq-1},$$

$$B(1) \longrightarrow B(2) \longrightarrow \Sigma^{p^2q-1}B(1),$$

$$B(2) \longrightarrow B(3) \longrightarrow \Sigma^{(p^2+p)q-1}B(2)$$

and $B(3) \equiv_p S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1\beta_1^{p-1}} e^{p^2q-1} \cup_{\beta_1} e^{(p^2+p)q-2} \cup_p e^{(p^2+p)q-1} \cup \ldots$.

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